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Sharp and Fuzzy Observables on Effect Algebras

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Abstract Observables on effect algebras and their fuzzy versions obtained by means of confidence measures (Markov kernels) are studied. It is shown that, on effect algebras with the (E)-property, given an observable and a confidence measure, there exists a fuzzy version of the observable. Ordering of observables according to their fuzzy properties is introduced, and some minimality conditions with respect to this ordering are found. Applications of some results of classical theory of experiments are considered.

Keywords Effect algebra · Observable · Hilbert space effects · PV-measure · POV-measure · Sufficient Markov kernel · Smearing

1 Introduction

In the frame of quantum mechanics, as a proper mathematical formulation of a physical quantity (so called observable), a normalized positive operator valued measure is considered, instead of the more traditional spectral measure (projection valued measure). This approach has also provided a frame to investigate imprecise measurements of a physical quantity. In the literature (e.g., [20]), the notion of a quantum mechanical fuzzy observable has been formulated as a smearing of a sharp observable (projection measure). In the present paper, we study smearing of observables in a more general frame of effect algebras. In analogy with [19, 20], we introduce the notion of a confidence measure (which is a Markov

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kernel), and we show that in a σ-orthocomplete effect algebra with an order determining set of σ-additive states which has the (E)-property [11], every confidence measure yields a smeared observable for a given (real) observable. We can then introduce a partial order for observables by putting $\xi \preceq \eta$ if η is a smearing of ξ; in this case we say that η is a *fuzzy version* of ξ [19]. If $\xi \preceq \eta$ and simultaneously, $\eta \preceq \xi$, we will write $\xi \sim \eta$, and say that ξ and η are *fuzzy equivalent*. In analogy with some recent papers [5], minimal elements in this ordering are called (postprocessing) *clean observables*, or *optimal measurements*, [19].

As a motivation, we give the following [20]. Let L be σ-orthocomplete effect algebra, $(\Omega, \mathcal{A})$ a measurable space, and $\xi : \mathcal{A} \to L$ a sharp observable on L, and m a σ-additive state on L. For every $E \in \mathcal{A}$, $\xi(E)$ is a sharp element of L (recall that $a \in L$ is sharp if 0 is the unique common lower bound of a and its orthosupplement a'), and $m(\xi(E)) = \int \delta_\omega(E) m(\xi(dx))$, where $\delta_\omega(E) = 1$ if $\omega \in E$, and $\delta_\omega(E) = 0$ if $\omega \notin E$. Since realistic measurements always have some imprecision, one may think that the points of Ω are to be replaced by probability distributions. If we replace the Dirac function δ_ω by a probability distribution $\nu_\omega : \mathcal{A} \to [0,1]$ in every point ω, we obtain $\int_\Omega \nu_\omega(E) m(\xi(d\omega))$, which is a smearing of $m(\xi(E))$. Under some appropriate additional assumptions on L (which are satisfied in the case of the effect algebra $\mathcal{E}(H)$ of the Hilbert space effects), there is a smeared observable η of ξ, such that $m(\eta(E)) = \int_\Omega \nu_\omega(E) m(\xi(d\omega))$ for every $E \in \mathcal{A}$ and every σ-additive state m.

2 Effect Algebras

An *effect algebra* [15] (see [16] and [22] for alternative definitions) is a set L with two distinguished elements 0, 1 and with a partial binary operation $\oplus : L \times L \to L$ such that for all $a, b, c \in L$ we have

(EAi) if $a \oplus b$ exists in L then $b \oplus a$ exists in L and $a \oplus b = b \oplus a$ (commutativity);
(EAii) if $b \oplus c$ exists in L and $a \oplus (b \oplus c)$ exists in L then $a \oplus b$ exists in L and $(a \oplus b) \oplus c$ exists in L, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(EAiii) for every $a \in L$ there is a unique $b \in L$ such that $a \oplus b = 1$ (orthosupplementation);
(EAiv) if $1 \oplus a$ is defined, then $a = 0$ (zero-one law).

As usual, we shall write $L = (L; \oplus, 0, 1)$ for effect algebras. If the assumptions of (EAii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

Let a, b be elements of an effect algebra L. We say that (i) a is *orthogonal to* b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than or equal to* b and write $a \le b$ iff there exists an element c in L such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \ge a$); b is the *orthosupplement* of a and write $b = a'$ iff b is the (unique) element in L such that $b \perp a$ and $a \oplus b = 1$. If $a \le b$, then the element c such that $a \oplus c = b$ is uniquely defined, and we write $c = b \ominus a$. In particular, for every $a \in L$, $a' = 1 \ominus a$, $a \perp b$ iff $b \le a'$, and $(a \oplus b)' = a' \ominus b$.

For a finite sequence $a_1, a_2, \ldots, a_n$, $n \ge 3$, we define recursively

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n, \tag{1}$$

supposing that $(a_1 \oplus \cdots \oplus a_{n-1})$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. Due to associativity of $\oplus$, the element (1) is correctly defined. Define $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then, due to commutativity and associativity of $\oplus$, for any permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ and any $0 \le k \le n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n}, \tag{2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n). \tag{3}$$

We say that a finite sequence $F = \{a_1, \ldots, a_n\}$ is *orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L, and we say that F *has the* $\oplus$*-sum* $\bigoplus F$, which is defined by

$$\bigoplus F = a_1 \oplus \cdots \oplus a_n. \tag{4}$$

An arbitrary system $G = \{a_i\}_{i\in I}$ of (not necessarily different) elements of L is said to be *orthogonal* if for any finite subset J of I, the system $\{a_i\}_{i\in J}$ is orthogonal. An orthogonal system $G = \{a_i\}_{i\in I}$ has an $\oplus$*-sum* in L, if in L there exists the join

$$\bigoplus_{i\in I} a_i := \bigvee_J \bigoplus_{i\in J} a_i, \tag{5}$$

where J runs over all finite subsets of I. In this case, we also write $\bigoplus G := \bigvee_J \bigoplus_{i\in I} a_i$.

Evidently, if $G = \{a_1, \ldots, a_n\}$ is orthogonal, then the $\oplus$-sums defined by (4) and (5) coincide.

Let $G = \{a_i\}_{i\in I}$ and $a_i = a$ for all $i \in I$. The greatest n such that $\bigoplus_{i\le n} a_i$ exists, is called the *isotropic index* of a. If $\bigoplus_{i\le n} a_i$ exists for all $n \in \mathbb{N}$, we say that the isotropic index of a is infinite. If $\bigoplus G$ exists and I is infinite, then $a = 0$. Indeed, let $a_0 = \bigoplus G$, then $a_0 = a_j \oplus \bigoplus_{i\in I\setminus\{j\}} a_i = a \oplus a_0$, which gives $a = 0$. Notice that if G is only orthogonal, then a is not necessarily 0.

We say that an effect algebra L is σ*-orthocomplete* (*orthocomplete*) if $\bigoplus_{i\in I} a_i$ exists for any countable (arbitrary) orthogonal system $\{a_i : i \in I\}$ of elements of L. We recall that an effect algebra is σ-orthocomplete iff for every nondecreasing sequence $\{a_i\}_{i\in\mathbb{N}}$ there is a supremum $a = \bigvee_{i\in\mathbb{N}} a_i$.

A mapping $s : L \to [0, 1]$ from L to unit interval $[0, 1]$ of real numbers is a *state* on L if (i) $s(1) = 1$, (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ exists in L. It is clear that $s(0) = 0$, and $s(a) \le s(b)$ whenever $a \le b$, $a, b \in L$. A state $s : L \to [0, 1]$ is said to be σ*-additive*, or *completely additive* if the equality

$$s\left(\bigoplus_{i\in I} a_i\right) = \sum_{i\in I} s(a_i), \tag{6}$$

holds for any countable, or arbitrary index set I, respectively, such that $\bigoplus_{i\in I} a_i$ exists in L.

A non-void system $\mathcal{S}$ of states on L is said to be *order determining*, if for $a, b \in L$, $a \le b$ iff $s(a) \le s(b)$ for all $s \in \mathcal{S}$. We denote by $\mathrm{Conv}(\mathcal{S})$ and $\mathrm{Conv}_\sigma(\mathcal{S})$ the convex and σ-convex hull of $\mathcal{S}$, respectively. Clearly, elements of $\mathrm{Conv}(\mathcal{S})$ and $\mathrm{Conv}_\sigma(\mathcal{S})$ are states on L, and moreover, $\mathcal{S}$ is order determining iff $\mathrm{Conv}(\mathcal{S})$ is order determining, or, equivalently, iff $\mathrm{Conv}_\sigma(\mathcal{S})$ is order determining.

Let L and P be effect algebras, a mapping $\phi : L \to P$ is a *morphism* if (i) $m(1_L) = 1_P$, where 1_L and 1_P are the unit elements in L and P, respectively, and (ii) $a \perp b$ implies $\phi(a) \perp \phi(b)$, and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. A morphism ϕ is called a σ*-morphism* (*complete morphism*) if it preserves all existing countable (arbitrary) $\oplus$-sums. A bijective morphism such that $a \perp b$ iff $\phi(a) \perp \phi(b)$, is an *isomorphism*. A σ-isomorphism, resp. complete isomorphism, is defined in an obvious way.

A subset P of an effect algebra L is a *sub-effect algebra*, if (i) $0 \in P$, $1 \in P$; (ii) $a, b \in P$, $a \perp b$ implies $a \oplus b \in P$, (iii) $a \in P$ implies $a' \in P$.

We recall that an effect algebra is:

– an *orthoalgebra* iff $a \perp a$ implies $a = 0$;
– an *orthomodular poset* iff $a \perp b$ implies $a \oplus b = a \vee b$ [26, 29];
– an *orthomodular lattice* iff it is a lattice ordered orthomodular poset;
– an *MV-effect algebra* iff it is lattice ordered and the equalities $(a \vee b) \ominus a = b \ominus (a \wedge b)$ are satisfied. We recall that MV-effect algebras coincide with *MV-algebras* introduced by Chang [8] as algebraic bases for many-valued logic.

Two of the most important prototypes of effect algebras are the following examples.

Example 2.1 Consider the closed interval [0, 1] of reals ordered by the natural way. For two numbers $a, b \in [0, 1]$ define $a \oplus b$ iff $a + b \leq 1$ and put then $a \oplus b = a + b$. Then [0, 1] is an orthocomplete effect algebra, and the effect algebra order coincides with the natural order of reals. With respect to this order, [0, 1] is a totally ordered, distributive lattice. We recall that $\{a_t\}$ is orthogonal iff $\sum_t a_t \leq 1$, and $\bigoplus_t a_t = \sum_t a_t$. There is only one state on [0, 1], namely the isomorphism $s_0(a) = a$. Clearly, s_0 is completely additive and the one-point set $\{s_0\}$ is order determining.

We recall that [0, 1] is also a prototype of MV-algebras.

Example 2.2 The set $\mathcal{E}(H)$ of all self-adjoint operators A on a Hilbert space H such that $0 \leq A \leq I$, where 0 is the zero and I the identity mapping, ordered by the usual order of self-adjoint operators, namely $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$. We define, on $\mathcal{E}(H)$, $A \perp B$ iff $A + B \leq I$, and then put $A \oplus B = A + B$. Then $(\mathcal{E}(H); \oplus, 0, I)$ becomes an effect algebra, in which the algebraic order coincides with the usual order that we started with. A system $\{A_t\}_t$ of elements from $\mathcal{E}(H)$ is orthogonal if $\sum_t A_t \leq I$, where the summation is in the weak, or equivalently in the strong operator topology, and then $\bigoplus_t A_t = \sum_t A_t$. The system $(\mathcal{E}(H); \oplus, 0, I)$ is an orthocomplete effect algebra which is not a lattice [17, 23].

Denote by $\mathcal{P}(H)$ the set of all orthogonal projections on H. Then $\mathcal{P}(H)$ is a sub-effect algebra of $\mathcal{E}(H)$, which is a complete orthomodular lattice.

We recall that $\mathcal{E}(H)$, as well as $\mathcal{P}(H)$, play an important role in the foundations of quantum mechanics and the theory of quantum measurements [2].

3 Observables on Effect Algebras

Let L be a σ-orthocomplete effect algebra, and $(\Omega, \mathcal{A})$ a measurable space. By an $(\Omega, \mathcal{A})$-*observable* on L we mean a mapping $\xi : \mathcal{A} \to L$ such that

(i) $\xi(\Omega) = 1$;
(ii) the system $\{\xi(E_i)\}_{i \in \mathbb{N}}$ is orthogonal and $\xi(\bigcup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} \xi(E_i)$ whenever $E_i \cap E_j = \emptyset$, $i \neq j$, and $E_i \in \mathcal{A}$ for $i \geq 1$.

If $(\Omega, \mathcal{A}) \subseteq (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then an observable $\xi : \mathcal{A} \to L$ is said to be a *real* observable.

Let $(\Omega_1, \mathcal{A}_1)$ be another measurable space, and let $f : \Omega \to \Omega_1$ be a measurable function such that $f^{-1}(A) \in \mathcal{A}$ whenever $A \in \mathcal{A}_1$. If $\xi : \mathcal{A} \to L$ is an observable, then $f \circ \xi : A \mapsto \xi(f^{-1}(A))$, $A \in \mathcal{A}_1$ is a $(\Omega_1, \mathcal{A}_1)$-observable on L. It is called the f*-function of* ξ. In particular, if ξ is a real observable on L and $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then $f \circ \xi$ is also a real observable on L.

If ξ is a $(\Omega, \mathcal{A})$-observable on L, and s is a σ-additive state on L, then $s_\xi := s \circ \xi : \mathcal{A} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{A})$. If ξ is a real observable, we denote by

$$s(\xi) := \int_{\mathbb{R}} t s_\xi(dt) \tag{7}$$

the *mean value* of ξ in s whenever the right-hand side of the above equation exists and is finite.

More generally, if $\xi : (X, \mathcal{A}) \to L$ is an observable, then for any Borel measurable function $f : X \to \mathbb{R}$, $f(\xi)$ is a real observable, and

$$\begin{aligned} s(f(\xi)) &= \int_{\mathbb{R}} us(f(\xi(du)) \\ &= \int_{\mathbb{R}} us(\xi(f^{-1}(du))) = \int_X f(t)s_\xi(dt)), \end{aligned}$$

using the integral transformation theorem.

The *spectrum* of a real observable ξ is the smallest closed subset C of $\mathbb{R}$ such that $\xi(C) = 1$.

For an $(\Omega, \mathcal{A})$-observable ξ on L, let $\mathcal{R}(\xi) := \{\xi(A) : A \in \mathcal{A}\}$ denote the range of ξ. Recall that an element $a \in L$ is called *sharp* if $a \wedge a' = 0$, that is, 0 is the only common lower bound of a and a'. Clearly, 0, 1 are sharp, and a is sharp iff a' is sharp. We will say that and observable ξ is *sharp* if its range consists of sharp elements.

Let us consider the following examples.

Example 3.1 Let H be a Hilbert space, and $\mathcal{E}(H)$ be the effect algebra of Example 2.2. Here the sharp elements coincide with projections. Indeed, if P is a projection, then $P = P^2$ implies $P \wedge (I - P) = P(I - P) = 0$, and conversely, for any $A \in \mathcal{E}(H)$, $0 \le A \le I$ implies that $A^{\frac{1}{2}}AA^{\frac{1}{2}} \le A^{\frac{1}{2}}IA^{\frac{1}{2}}$, which yields $0 \le A^2 \le A$. Then $0 \le A - A^2 \le A$, and $(I - A) - (A - A^2) = (I - A)^2 \ge 0$ yields $I - A \ge A - A^2$, hence $A - A^2$ is a common lower bound of A and $I - A$. Hence, A is sharp iff $A = A^2$, equivalently, iff A is a projection. Sharp observables on $\mathcal{E}(H)$ are then exactly those whose ranges are in $\mathcal{P}(H)$. These observables are called *projection valued observables* (PV-observables, in short), while general observables are called *positive operator valued observables* (POV-observables, in short). Owing to spectral theorem, real (bounded) PV-observables are in one-to-one correspondence with (bounded) self-adjoint operators.

Example 3.2 Let X be a nonempty set. A *tribe* over X is a collection of functions $\mathcal{T} \subseteq [0, 1]^X$ such that the zero function $\underline{0}(x) = 0$ is in $\mathcal{T}$ and the following is satisfied:

(T1) $f \in \mathcal{T} \Longrightarrow 1 - f \in \mathcal{T}$;
(T2) $f, g \in \mathcal{T} \Longrightarrow f \dot{+} g := \min(f + g, 1) \in \mathcal{T}$;
(T3) $f_n \in \mathcal{T}, n \in \mathbb{N}$ and $f_n \nearrow f$ (pointwise) $\Longrightarrow f \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called *fuzzy sets* or *fuzzy events*. Sharp elements in $\mathcal{T}$ coincide with the characteristic functions contained in $\mathcal{T}$. We put $\mathcal{B}(\mathcal{T}) := \{B \subseteq X : \chi_B \in \mathcal{T}\}$, where χ_B is the characteristic function of the set B. Then $\mathcal{B}(\mathcal{T})$ is a σ-algebra of sets, which is isomorphic with the system of all sharp elements of $\mathcal{T}$. The restriction of any σ-additive state on $\mathcal{T}$ to $\mathcal{B}(\mathcal{T})$ is a probability measure. Due to Butnariu and Klement theorem [6], every element in $\mathcal{T}$ is a measurable function with respect to $\mathcal{B}(\mathcal{T})$. Moreover, every σ-additive state m on $\mathcal{T}$ has an integral representation

$$m(f) = \int_X f dP, \tag{8}$$

where P is the restriction of m to $\mathcal{B}(\mathcal{T})$, i.e., $P(A) = m(\chi_A)$.

Let ξ be an $(\Omega, \mathcal{A})$-observable on $\mathcal{T}$. Define $\nu : X \times \mathcal{A} \to [0, 1]$, $\nu(x, A) = \xi(A)(x)$, where $\xi(A) \in \mathcal{T}$. The mapping ν has the following properties:

(c1) for any fixed $x \in X$, $\nu(x, .)$ is a probability measure on $\mathcal{A}$;
(c2) for any fixed $A \in \mathcal{A}$, $\nu(., A)$ belongs to $\mathcal{T}$.

Conversely, every mapping $\nu : X \times \mathcal{A} \to [0, 1]$ with properties (c1), (c2) gives rise to an observable on $\mathcal{T}$ given by $\xi(A) = \nu(., A)$.

Clearly, an observable is sharp if its range consists of characteristic functions from $\mathcal{T}$. In fact, if π is a sharp $(\Omega, \mathcal{A})$-observable on $\mathcal{T}$, then π is a σ-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{T})$, therefore there is an $(\mathcal{A}, \mathcal{B}(\mathcal{T}))$-measurable function $g : X \to \Omega$ such that $\pi(A) = g^{-1}(A)$, $A \in \mathcal{A}$, and $\nu(x, A) = \chi_{g^{-1}(A)}(x)$ [29].

Example 3.3 Recall that an MV-algebra can be defined as a system $(M, \dot{+}, *, 0, 1)$ consisting of a nonempty set M, two constants 0 and 1, a unary operation * and a binary operation $\dot{+}$ satisfying the following axioms:

(MV1) $a \dot{+} b = b \dot{+} a$;
(MV2) $a \dot{+} (b \dot{+} c) = (a \dot{+} b) \dot{+} c$;
(MV3) $a \dot{+} a^* = 1$;
(MV4) $a \dot{+} 0 = a$;
(MV5) $a^{**} = a$;
(MV6) $0^* = 1$;
(MV7) $a \dot{+} 1 = 1$;
(MV8) $(a^* \dot{+} b)^* \dot{+} b = (a \dot{+} b^*)^* \dot{+} a$.

The above axioms are equivalent with the original axioms introduced by Chang in [8] (see [7]). A partial order can be introduced on M by putting $a \le b$ iff $a^* \dot{+} b = 1$. With respect to this ordering, M becomes a distributive lattice, where $a \vee b = (a^* \dot{+} b)^* \dot{+} b$, $a \wedge b = (a^* \vee b^*)^*$. By putting $a \oplus b = a \dot{+} b$ iff $a \le b^*$, we obtain an effect algebra $(M; \oplus, 0, 1)$, where a^* is the orthosupplement of a for all $a \in M$. Conversely, an effect algebra $(L; \oplus, 0, 1)$ can be organized into an MV-algebra (i.e., it is an MV-effect algebra) iff L is a lattice, and for any $a, b \in L$, the equality $(a \vee b) \ominus a = b \ominus (a \wedge b)$ holds. The total operation $\dot{+}$ is defined by $a \dot{+} b = (a \oplus (a' \wedge b))$, and $a^* = a'$ [9]. An MV-effect algebra M is σ-orthocomplete (σ-MV algebra), or orthocomplete (complete MV algebra) iff M is a σ-lattice, or a complete lattice, respectively. Sharp elements in an MV-algebra M coincide with the idempotents in M, that is, $a \wedge a^* = 0$ iff $a \dot{+} a = a$. The set $\mathcal{B}(M)$ of sharp elements of M forms a Boolean subalgebra of M. If M is σ-complete, then $\mathcal{B}(M)$ is a Boolean σ-algebra [13].

Every tribe is a σ-MV algebra with $f \dot{+} g = \min(f + g, 1)$, $f^* = 1 - f$, and where the lattice operations $\vee, \wedge$ coincide with pointwise supremum and infimum, respectively, of [0, 1]-valued functions on X.

By the Loomis–Sikorski theorem for σ-MV algebras [1, 12, 25], to every σ-MV algebra there is a triple $(X, \mathcal{T}, h)$ consisting of a tribe $\mathcal{T}$ of fuzzy sets on a nonvoid set X and a surjective σ-homomorphism (of σ-MV-algebras) $h : \mathcal{T} \to M$, such that the restriction of h to $\mathcal{B}(\mathcal{T})$ maps the latter set onto $\mathcal{B}(M)$.

Let M be a σ-MV-effect algebra, and let $(X, \mathcal{T}, h)$ be its representation by the Loomis–Sikorski theorem. Let ξ be an $(\Omega, \mathcal{A})$ observable on M. For every $A \in \mathcal{A}$, there is an $f_A \in \mathcal{T}$ such that $h(f_A) = \xi(A)$, where f_A is unique up to h-null sets. Define $\nu : X \times \mathcal{A} \to [0, 1]$ by putting $\nu(x, A) = f_A(x)$. Clearly, for a fixed $A \in \mathcal{A}$, $\nu_A \in \mathcal{T}$. Moreover, $\xi(A) = h(\nu_A)$. Let $\{E_i\}_i$ be a disjoint sequence of elements of $\mathcal{A}$, and put $E = \bigcup_i E_i$. Then $\xi(E) = \bigoplus_i \xi(E_i)$. Choose functions $f, f_i, i = 1, 2, \dots$ in $\mathcal{T}$ such that $h(f) = \xi(E)$, $h(f_i) = \xi(E_i)$, $i = 1, 2, \dots$. Then we have $h(f) = \bigoplus h(f_i) = h(\min(\sum_{i=1}^{\infty} f_i, 1))$. Consider an orthogonal sequence $g_i, i = 1, 2, \dots$, where $g_1 = f_1$, and for $i \ge 1$, $g_i = f_i \wedge (g_1 + \cdots + g_{i-1})^*$. We have $h(f_1) = h(g_1)$, and assume that $h(g_i) = h(f_i)$ for $i < k$. Then $h(g_k) = h(f_k \wedge (g_1 + \cdots + g_{k-1})^*) = h(f_k) \wedge h(g_1 + \cdots + g_{k-1})' = h(f_k) \wedge (h(f_1) \oplus \cdots \oplus h(f_{k-1}))' = h(f_k)$. We proved, by induction, that $h(f_i) = h(g_i), i = 1, 2, \ldots$, which entails that $h(\{x : f_i(x) \neq g_i(x)\}) = 0,\ i = 1, 2, \ldots$ (we identify sets with their characteristic functions). Clearly, $\sum_{i=1}^{\infty} f_i > 1$ iff $f_i \neq g_i$ for at least one i, therefore $\{x : \sum_{i=1}^{\infty} f_i > 1\} = \bigcup_{i=1}^{\infty}\{x : f_i \neq g_i\} \in \ker(h)$, and this entails that $h(\{x : f(x) \neq \sum_{i=1}^{\infty} f_i(x)\}) = 0$. This shows that $\nu(x, E) = \sum_{i=1}^{\infty} \nu(x, E_i)$ for all x up to an h-null set.

4 Smearing of Observables

4.1 Markov Kernels

Let L be a σ-orthocomplete effect algebra with a system $\mathcal{S}$ of σ-additive states, and let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. Let ξ be an $(X, \mathcal{F})$-observable on L. Consider a mapping $\nu : X \times \mathcal{G} \to [0, 1]$ with the following properties:

(i) for any fixed $x \in X$, $\nu_x(.) := \nu(x, .) : \mathcal{G} \to [0, 1]$ is a probability measure;
(ii) for any fixed $G \in \mathcal{G}$, the mapping $x \mapsto \nu_G(x) := \nu(x, G)$ is $\mathcal{F}$-measurable.

That is, ν is a *Markov kernel* (we note that in analogy with [20], ν may be called also a *confidence measure*). Let $m \in \mathcal{S}$. The integral

$$\int_X \nu_G(x) m(\xi(dx))$$

converges by the dominating convergence theorem. If there is an observable $\eta : (Y, \mathcal{G}) \to L$ such that

$$m(\eta(G)) = \int_X \nu_G(x) m(\xi(dx)) \tag{9}$$

for every $m \in \mathcal{S}$, the we will call η a *fuzzy version* of ξ, or a *smearing of* ξ *in the states* $m \in \mathcal{S}$. If moreover the system $\mathcal{S}$ is order determining, then the equations (9) uniquely determine η, and we call η simply a fuzzy version (smearing) of ξ. In this case, we will write $\xi \preceq \eta$. If equation (9) holds for every $m \in \mathcal{S}$, we will write symbolically

$$\eta(G) = \int_X \nu_G(x) \xi(dx). \tag{10}$$

The relation $\preceq$ is reflexive, since the mapping $(x, G) \mapsto \delta_x(G) = \chi_G(x)$ is a Markov kernel, and $\xi(G) = \int \chi_G(x)\xi(dx)$. It is also transitive. Indeed, let $\xi \preceq \eta$ and $\eta \preceq \zeta$, where $\xi : (X, \mathcal{F}) \to L,\ \eta : (Y, \mathcal{G}) \to L,\ \zeta : (Z, \mathcal{H}) \to L,\ \eta(G) = \int_X \nu_1(x, G)\xi(dx),\ \zeta(H) = \int_Y \nu_2(y, H)\eta(dy) = \int_Y \nu_2(y, H) \int_X \nu_1(x, dy)\xi(dx)$. It is well known that

$$\nu_3(x, H) := \int_Y \nu_2(y, H)\nu_1(x, dy) \tag{11}$$

is a Markov kernel (see [19] for a detailed proof). Therefore, $\preceq$ is a preorder, and it can be made a partial order in the usual way. If $\xi \preceq \eta$, and $\eta \preceq \xi$, we will write $\xi \sim \eta$, and we will say that ξ and η are *fuzzy equivalent*. Obviously, ξ is a minimal element with respect to $\preceq$ if $\eta \preceq \xi$ implies $\eta \sim \xi$. Minimal observables are called *clean* in accordance with [5]. We note that in [19], the relation $\preceq$ is defined in the opposite direction, and maximal elements are called optimal measurements.

4.2 Weak Markov Kernels

Let $(\Omega, \mathcal{A})$ be a measurable space. Then $M_1^+(\Omega, \mathcal{A})$ will denote the set of all probability measures on $(\Omega, \mathcal{A})$.

The notion of a Markov kernel can be weakened as follows. Let $(\Omega, \mathcal{A})$ and $(\Omega_1, \mathcal{A}_1)$ be measurable spaces. Let $\mathcal{P} \subseteq M_1^+(\Omega, \mathcal{A})$, and let $\nu : \Omega \times \mathcal{A}_1 \to \mathbb{R}$. We will say that ν is a *weak Markov kernel with respect to* $\mathcal{P}$ if

(i) $\omega \mapsto \nu(\omega, B)$ is $\mathcal{A}$-measurable for all $B \in \mathcal{A}_1$;
(ii) for every $B \in \mathcal{A}_1$, $0 \le \nu(\omega, B) \le 1$, $\mathcal{P}$-a.e.;
(iii) $\nu(\omega, \Omega_1) = 1$, $\mathcal{P}$-a.e. and $\nu(\omega, \emptyset) = 0$, $\mathcal{P}$-a.e.
(iv) if $\{B_n\}$ is a sequence in $\mathcal{A}_1$ such that $B_n \cap B_m = \emptyset$ for $m \neq n$, then

$$\nu\left(\omega, \bigcup_n B_n\right) = \sum_n \nu(\omega, B_n), \mathcal{P} - a.e.$$

Note that a weak Markov kernel with respect to the whole $M_1^+(\Omega, \mathcal{A})$ is in fact a Markov kernel.

It is easy to see that if ν is a weak Markov kernel with respect to $\mathcal{P}$, then

$$\nu(P)(B) := \int_\Omega \nu(\omega, B) P(d\omega), B \in \mathcal{A}_1 \tag{12}$$

is a probability measure on $\mathcal{A}_1$ for all probability measures $P \in \mathcal{P}$.

Let L be a σ-orthocomplete effect algebra with an order determining system of σ-additive states $\mathcal{S}$, and let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. Let ξ be a $(X, \mathcal{F})$-observable on L. If $\nu : X \times \mathcal{G} \to \mathbb{R}$ is a weak Markov kernel with respect to $\mathcal{P} = \{m \circ \xi : m \in \mathcal{S}\}$, then

$$\nu(m \circ \xi)(B) = \int_X \nu(\omega, B) m \circ \xi(dx)$$

is a probability measure on $(Y, \mathcal{G})$, and if there is an observable η on L such that

$$\nu(m \circ \xi)(B) = m(\eta(B)),$$

for all $B \in \mathcal{G}$ and all $m \in \mathcal{S}$, then we will also call η a *fuzzy version* (or a *smearing*) of ξ (in the states $m \in \mathcal{S}$, if the latter set is not order determining). If $\mathcal{S}$ is order determining, we also write $\xi \preceq \eta$.

Remark 4.1 We note that a weak Markov kernel $\nu : X \times \mathcal{G} \to [0, 1]$ (with respect to one probability measure P) is called a *random measure* in the literature. If $\mathcal{G}$ is the Borel σ-algebra of subsets of a complete separable metric space Y, then there exists a regular version ν^* of ν, such that ν^* is a Markov kernel, and

$$\forall G \in \mathcal{G}, \nu(x, G) = \nu^*(x, G), a.e.P \tag{13}$$

(see, e.g. [27, VI.1. 21.]). For a more general version, see Theorem 6.3.

Let L be a σ-orthocomplete effect algebra with an order determining set of σ additive states $\mathcal{S}$, ξ be an $(X, \mathcal{F})$-observable on L, and $(Y, \mathcal{G})$ be a complete metric space with the Borel σ-algebra. Let $\nu : X \times \mathcal{G} \to [0, 1]$ be a weak Markov kernel with respect to $\mathcal{P} = \{m \circ \xi : m \in \mathcal{S}\}$. Then for every $m \in \mathcal{S}$ there exists a Markov kernel ν_m^*.

If, in addition, there is a faithful state[1] m_0 on L, then the regular version $\nu^*_{m_0}$ is the regular version of ν for all $m \in \mathcal{S}$. This will follow from Theorem 6.3.

Example 4.2 1. We can see that in Examples 3.2 and 3.3 Markov kernels are closely related to observables. Namely, in Example 3.2 observables coincide with certain Markov kernels.

In Example 3.3, we have the following situation. Let $(X, \mathcal{T}, h)$ be the Loomis–Sikorski representation of a σ-MV-algebra M. Let $\xi : (\Omega, \mathcal{A}) \to M$ be an observable. For every $A \in \mathcal{A}$, choose $f_A \in \mathcal{T}$ such that $\xi(A) = h(f_A)$ (for definiteness, we may choose the (unique) continuous function in the corresponding class), and define $\nu(x, A) = f_A(x)$, then $\nu : X \times \mathcal{A} \to [0, 1]$ is a weak Markov kernel with respect to the family $\mathcal{P} = \{m \circ h\}$, m a σ-additive state on M, of probability measures on $\mathcal{B}(\mathcal{T})$. Owing to Butnariu–Klement theorem we have

$$m(\xi(A)) = m(h(f_A)) = \int_X f_A(x)P(dx),$$

where $P = m \circ h/\mathcal{B}(\mathcal{T}) = m \circ (h/\mathcal{B}(\mathcal{T}))$. The restriction $h/\mathcal{B}(\mathcal{T}) : \mathcal{B}(\mathcal{T}) \to \mathcal{B}(M)$ can be considered as a sharp observable on M, and any other observable may be considered as a smearing of it in all σ-additive states on M.

2. Let η, ξ be real observables on an effect algebra L such that $\eta = f \circ \xi$ for some Borel function $f : \mathbb{R} \to \mathbb{R}$. Then for every σ-additive state m on L, $m(\eta(E)) = m(\xi(f^{-1}(E)) = \int \chi_{f^{-1}(E)}(\lambda)m(\xi(d\lambda))$, $E \in \mathcal{B}(\mathbb{R})$. Put $\nu(\lambda, E) = \chi_{f^{-1}(E)}(\lambda)$. It is easy to see that ν is a Markov kernel, and η is a smearing of ξ. However, if ξ is sharp, then η is sharp, too. Hence a smearing of a sharp observable may be sharp as well.

4.3 POV-Measure with Commuting Range

In this section, we keep our standing assumption that L is a σ-orthocomplete effect algebra, and S is an order determining set of σ-additive states on L.

Let $\xi : (X, \mathcal{A}) \to L$ and $\eta : (Y, \mathcal{B}) \to L$ be observables on L. Assume that η is a smearing of ξ, hence there is a (weak) Markov kernel $\nu(x, E) : X \times \mathcal{B} \to [0, 1]$ such that for every $m \in S$, and $E \in \mathcal{B}$,

$$m(\eta(E)) = \int_X \nu_E(x)m(\xi(dx)).$$

For E fixed, $\nu_E : X \to [0, 1]$ is a measurable function, and the right hand side of the above equality is a mean value of the observable $\nu_E(\xi)$ in the state m, so that for every σ-additive state m,

$$m(\eta(E)) = m(\nu_E(\xi)).$$

Define the observable $\Lambda_{\eta(E)}$ by $\Lambda_{\eta(E)}\{1\} = \eta(E)$, $\Lambda_{\eta(E)}\{0\} = \eta(E)'$, then we obtain, for all $m \in S$,

$$m(\Lambda_{\eta(E)}) = m(\nu_E(\xi)).$$

[1] We recall that a state m_0 on an effect algebra L is *faithful* if $m_0(a) = 0 \Longrightarrow a = 0$. Clearly, for every state m on L, $m_0(a) = 0 \implies m(a) = 0$ $(a \in L)$, whence for every observable ξ it holds $m \circ \xi \ll m_0 \circ \xi$. For example, if H is a complex, separable Hilbert space, then there exists a faithful state m_0 on $\mathcal{E}(H)$.

Theorem 4.3 *Let $\xi : (X, \mathcal{A}) \to L$ and $\eta : (Y, \mathcal{B}) \to L$ be observables on L, such that η is a smearing of ξ with a (weak) Markov kernel ν such that for all Borel sets E, $\nu(x, E) \in \{0, 1\}$ a.e. $m \circ \xi$, $m \in S$. Then $\mathcal{R}(\eta) \subseteq \mathcal{R}(\xi)$.*

If η and ξ are sharp, and η is real then also the converse statement is true.

Proof Let $E \in \mathcal{B}$. Under the hypotheses, we have for every $m \in S$,

$$\begin{aligned} m(\eta(E)) &= \int \nu(x, E) m(\xi(dx)) \\ &= \int_{\{x:\nu(x,E)=1\}} \nu(x, E) m(\xi(dx)) \\ &= m(\xi(\{x : \nu(x, E) = 1\})). \end{aligned}$$

Since S is order determining, we obtain $\eta(E) = \xi(\nu_E)^{-1}(\{1\}) \in \mathcal{R}(\xi)$.

We note that in this case we have $\Lambda_{\eta(E)} = \nu_E(\xi)$.

Let both η and ξ be sharp, and assume that $\mathcal{R}(\eta) \subseteq \mathcal{R}(\xi)$. Since the range of a sharp observable is a Boolean σ-algebra [14], if η is real, we can apply [29, Theorem 1.4], to obtain that there is a measurable function $f : X \to \mathbb{R}$ such that

$$\eta(E) = \xi(f^{-1}(E))$$

for all Borel sets $E \subseteq \mathbb{R}$, and the function f is unique up to a ξ-null set. Putting $\nu(x, E) = \chi_{f^{-1}(E)}$, we obtain

$$m(\eta(E)) = m(\xi(f^{-1}(E)) = \int \nu(x, E) m(\xi(dx))$$

for all states $m \in S$. Hence η is a smearing of ξ with a Markov kernel $\nu(x, E) = \chi_{f^{-1}(E)}(x) \in \{0, 1\}$. □

For different versions of the next theorem see [30, 31].

Theorem 4.4 *On the effect algebra $\mathcal{E}(H)$ of a separable H, an observable (POV-measure) is a smearing of a sharp observable (PV-measure) if and only if the range $\mathcal{R}(\eta)$ of η consists of mutually commuting effects.*

Moreover, a sharp real observable η is a smearing of a sharp observable ξ if and only if $\mathcal{R}(\eta) \subseteq \mathcal{R}(\xi)$, equivalently, if and only if η is a function of ξ.

Proof Let $\eta : (Y, \mathcal{B}) \to \mathcal{E}(H)$ be a POV-measure that is a smearing of a PV-measure $\xi : (X, \mathcal{A}) \to \mathcal{E}(H)$. Then for every set E, and every state m, $m(\eta(E)) = \int_X \nu(x, E) m(\xi(dx)) = m(\nu_E(\xi))$, which implies that $\eta(E) = \nu_E(\xi)$, where $\nu_E(\xi)$ is a function of ξ. It follows that all the spectral projections of the self-adjoint operator $\eta(E)$ belong to $\mathcal{R}(\xi)$, and this implies that $\mathcal{R}(\eta)$ consists of mutually commuting effects.

Conversely, let the range of a POV measure η consist of commuting effects. By well known von Neumann theorem (see also [29]), there exists a self-adjoint operator V and Borel measurable functions f_E such that $\eta(E) = f_E(V)$. Then we have, for every state m,

$$m(\eta(E)) = \int_X f_E(x) m(P^V(dx)),$$

where P^V is the spectral measure of V. Define $\nu(x, E) := f_E(x)$. We will show that ν is a weak Markov kernel.

(i) Since $0 \le \eta(E) \le 1$, $0 \le f_E(x) \le 1$ on the spectrum of V, hence $0 \le f_E \le 1$ a.e. $m \circ P^V$ for all m.
(ii) $f_Y(V) = \eta(Y) = 1$ implies that $\int_X \nu(x, Y) m(P^V(dx)) = 1$, and as $0 \le \nu(x, Y) \le 1$, we get $\nu(x, Y) = 1$ a.e. $m \circ P^V$ for all m. Similarly we show that $\nu(x, \emptyset) = 0$ a.e. $m \circ P^V$ for all m.
(iii) Let $E = \bigcup_{i=1}^{\infty} E_i$, where $E_i \cap E_j = \emptyset$ whenever $i \neq j$. From

$$\eta(E) = \sum_{i=1}^{\infty} \eta(E_i) \tag{14}$$

(the convergence in weak sense), we obtain that

$$\int \nu(x, E) P^V(dx) = \sum_{i=1}^{\infty} \int \nu(x, E_i) P^V(dx). \tag{15}$$

Moreover, for every n, $\eta(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \eta(E_i)$ entails $f_{\bigcup_{i=1}^n E_i}(V) = \sum_{i=1}^n f_{E_i}(V) = (\sum_{i=1}^n f_{E_i})(V)$, which entails that $f_{\bigcup_{i=1}^n}(x) = \sum_{i=1}^n f_{E_i}(x)$ on the spectrum of V. From this we derive that $0 \le \nu(x, \bigcup_{i=1}^n E_i) = \sum_{i=1}^n \nu(x, E_i) \le 1$ for all n. Therefore $\sum_{i=1}^{\infty} \nu(x, E_i)$ exists, and (14) yields that

$$f_E(V) = \sum_{i=1}^{\infty} f_{E_i}(V),$$

whence $f_E(x) = \sum_{i=1}^{\infty} f_{E_i}(x)$ on the spectrum of V. We conclude that $\nu(x, E) = \sum_{i=1}^{\infty} E_i$ a.e. $m \circ P^V$ for every state m. This concludes the proof that ν is a weak Markov kernel, and η is a smearing of $\xi := P^V$.

Let both ξ and η be sharp, and let η be real. If $\mathcal{R}(\eta) \subseteq \mathcal{R}(\xi)$, Theorem 4.3 implies that η is a smearing of ξ.

Conversely, assume that η is a smearing of ξ with a (weak) Markov kernel ν. For every state m, and every $E \in \mathcal{B}(\mathbb{R})$, $m(\eta(E)) = \int \nu(x, E) m(\xi(dx)) = m(\nu_E(\xi))$. We may also write $m(\Lambda_{\eta(E)}) = m(\nu_E(\xi))$ for every m, where $\Lambda_{\eta(E)}$ is the $0-1$ observable associated with $\eta(E)$, which yields $\Lambda_{\eta(E)} = \nu_E(\xi)$. Then $\eta(E) = \lambda_{\eta(E)}\{1\} = \xi(\nu_E^{-1}\{1\}) \in \mathcal{R}(\xi)$. We obtained that $\mathcal{R}(\eta) \subseteq \mathcal{R}(\xi)$, equivalently, that $\eta = f(\xi)$ for a measurable function f. □

4.4 Some Examples of Minimal Observables

Example 4.5 Let L be any σ-orthocomplete effect algebra, and let $a_1, a_2, \dots, a_n$ be elements of L such that $\bigoplus_{i \le n} a_i = 1$. Choose real numbers $r_1, r_2, \dots, r_n$. Then we may construct a (real) observable ξ on L by putting $\xi(E) = \bigoplus_{\{i : r_i \in E\}} a_i$, $E \in \mathcal{B}(\mathbb{R})$. We clearly have $\xi(\{r_i\}) = a_i$, $i \le n$, and $\{r_1, r_2, \dots, r_n\}$ is the spectrum of ξ.

Now let $L = \mathcal{E}(H)$, where H is a finite dimensional Hilbert space. Let $A_1, A_2, \dots, A_n$ be effects in $\mathcal{E}(H)$ such that $\sum_{i \le n} A_i = 1$. That is, $A_1, A_2, \dots, A_n$ is a resolution of unity in $\mathcal{E}(H)$. Let η be a real observable on $\mathcal{E}(H)$ such that $\eta(E) = \sum_{\{i : \alpha_i \in E\}} A_i$, $E \in \mathcal{B}(\mathbb{R})$, where α_i, $i \le n$ are real numbers. Clearly, $\eta(\alpha_i) = A_i$, $i \le n$, and $\{\alpha_i : i \le n\}$ is the spectrum of η. Notice that every POV measure η on a finite dimensional Hilbert space H is of this type, and $\eta(\{\alpha_i\})$, $i \le n$, are atoms of the range $\mathcal{R}(\eta)$ of the observable η.

Since every $A_i, i \leq n$ is a self adjoint operator on H, it has a spectral decomposition $A_i = \sum_{j=1}^{k_i} a_{i_j} P_{i_j}$, where P_{i_j}'s are one dimensional projections with $\sum_{j=1}^{k_i} P_{i_j} = 1$, and $0 \leq a_{i_j} \leq 1$ are eigenvalues of A_i (not necessarily all different). The elements $B_{i_j} := a_{i_j} P_{i_j}$ are effects in $\mathcal{E}(H)$. Owing to $\sum_{i\leq n} A_i = 1$, we have $\sum_{i\leq n}\sum_{j=1}^{k_i} B_{i_j} = 1$. By the first paragraph, we may choose real numbers $(\beta_{i_j})_{i_j}$ and construct an observable ξ such that $\xi(\beta_{i_j}) = B_{i_j}$, and more generally, $\xi(E) = \sum_{\{i_j:\beta_{i_j}\in E\}} B_{i_j}$, $E \in \mathcal{B}(\mathbb{R})$.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(\beta_{i_j}) = \alpha_i$, $i = 1, \ldots, n$, $j = 1, \ldots, k_i$ and $f(r) = 0$ if $r \neq \beta_{i_j}$ for all i_j. Since the range of f is finite, it is measurable. Moreover, $f^{-1}(\alpha_i) = \{\beta_{i_j}, j = 1, 2, \ldots, k_i\}$. Therefore, $\eta(\alpha_i) = A_i = \sum_{j=1}^{k_i} B_{i_j} = \sum_{j=1}^{k_i} \xi(\beta_{i_j}) = \xi(f^{-1}(\alpha_i))$. Hence $\eta = f \circ \xi$, and hence η is a smearing of ξ.

We have the following conclusion: if the range of a POV observable η contains atoms with rank greater than 1, then there is an observable ξ with $\xi \preceq \eta$. It follows that η is not minimal. The converse statement is also proved in [5]. Now we will rewrite it in our setting.

Assume that η is a POV measure with the spectrum $\{y_j\}_{j\leq n}$ such that $\eta(y_j)$ for every $j \leq n$ is an effect of rank one, that is, a multiple of a one-dimensional projection. Assume that η is a smearing of a POV ξ. This entails, for every j,

$$\eta(y_j) = \sum_i \nu(x_i, y_j)\xi(x_i) = \sum_{i\in\ell(j)} \nu(x_i, y_j)\xi(x_i),$$

where $i \leq m$ for some $m \in \mathbb{N}$ and we put, for every j, $\ell(j) = \{i : \nu(x_i, y_j) \neq 0\}$. Observe that for every i, $\sum_j \nu(x_i, y_j) = 1$, since $\nu(x_i, .)$ is a probability measure.

Since $\eta(y_j)$ is rank one, we have $\xi(x_i) = \beta_i\eta(y_j)$ $\forall i \in \ell(j)$, with $0 < \beta_i \leq 1$. This yields

$$\eta(y_j) = \sum_{i\in\ell(j)} \nu(x_i, y_j)\beta_i\eta(y_j). \tag{16}$$

Define $\alpha_i^j := \nu(x_i, y_j)\beta_i$, then (16) implies $\sum_{i\in\ell(j)} \alpha_i^j = 1$, $\alpha_i^j\eta(y_j) = \nu(x_i, y_j)\xi(x_i)$, and from

$$\sum_j \nu(x_i, y_j)\xi(x_i) = \xi(x_i)$$

we get, putting $\bar{\nu}(y_j, x_i) := \alpha_i^j$, if $i \in \ell(j)$, $\bar{\nu}(y_j, x_i) := 0$ otherwise,

$$\xi(x_i) = \sum_j \bar{\nu}(y_j, x_i)\eta(y_j), \tag{17}$$

which shows that ξ is a smearing of η. This shows that $\xi \preceq \eta$ implies $\eta \sim \xi$, i.e. η is minimal.

Example 4.6 Let L be a σ-orthocomplete effect algebra, $(\Omega, \mathcal{A})$ a measurable space. Let $\mathcal{O}(\Omega, \mathcal{A}, L)$ denote the set of all observables on L with the value space $(\Omega, \mathcal{A})$. Let $\mathcal{O} \subseteq \mathcal{O}(\Omega, \mathcal{A}, L)$. We say that an observable ξ is *minimal in* $\mathcal{O}$ (or $\mathcal{O}$*-clean*), if for any $\eta \in \mathcal{O}$, the condition $\eta \preceq \xi$ implies that $\eta \sim \xi$.

Recall that any element a in an effect algebra L defines an observable ξ^a with the outcome space $\Omega = \{0, 1\}$ by

$$\xi^a(\{1\}) = a, \qquad \xi^a(\{0\}) = a'.$$

Observables of this type are called *1-0-observables*.

In [19], minimality in the class $\mathcal{O}(\{0,1\}, H)$ of the 1-0 observables on the effect algebra $\mathcal{E}(H)$ is considered. In accordance with [19], the 1-0 observable corresponding to $A \in \mathcal{E}(H)$ will be denoted by E^A. We recall that the set $\mathcal{O}(\{0,1\}, H)$ is convex, and

$$\lambda E^A + (1-\lambda)E^B = E^{\lambda A + (1-\lambda)B}$$

whenever $\lambda \in [0, 1]$.

Proposition 4.7 [19] *Let $A, B \in \mathcal{E}(H)$ and let E^A, E^B be the corresponding* 1-0 *observables. Then $E^B \preceq E^A$ if and only if there are numbers $s, t \in [0, 1]$ such that $A = tB + sB'$.*

As a consequence of Proposition 4.7 we obtain that $E^A \sim E^B$ iff $A = B$ or $A = B'$.

Proposition 4.8 [19, Proposition 3] *Let $A \in \mathcal{E}(H)$. The observable E^A is minimal in $\mathcal{O}(\{0,1\}, H)$ if and only if $\|A\| = \|A'\| = 1$.*

We note that the extreme elements in the convex set $\mathcal{E}(H)$ are projection operators [10, Lemma 2.3]. According to Proposition 4.8, for every projection P, the corresponding 1-0 observable E^P is minimal in $\mathcal{O}(\{0,1\}, H)$, but there are also other minimal observables. However, in accordance with Example 4.5, if $\dim H \geq 3$, then the observable E^A for any effect A is not minimal in the set of all real observables on $\mathcal{E}(H)$.

5 Smearings of Observables on Effect Algebras with the (E)-Property

Notice that if $\mathcal{S}$ is an order determining system of states on an effect algebra L, then by replacing $\mathcal{S}$ by its (σ)-convex hull $\mathrm{Conv}(\mathcal{S})$, we may always assume that $\mathcal{S}$ is a (σ)-convex set.

Let L be a σ-orthocomplete effect algebra with an order determining system $\mathcal{S}$ of σ-additive states, $(X, \mathcal{F})$ be a measurable space. Every observable $\xi : \mathcal{F} \to L$ can be characterized by a mapping $\Phi_\xi : \mathcal{S} \to M_1^+(X, \mathcal{F})$ defined by

$$\Phi_\xi(m)(F) = m \circ \xi(F), \quad m \in \mathcal{S}, F \in \mathcal{F}. \tag{18}$$

Here $F \to m \circ \xi(F) = \Phi_\xi(m)(F)$, $F \in \mathcal{F}$ is the probability distribution of the observable ξ in the state m.

We will try to find conditions under which to given observable and Markov kernel there exists a fuzzy version. For the sake of simplicity, we will concentrate to real observables.

The following definitions were introduced in [11]. Let $S \neq \emptyset$ be a convex set. A mapping $f : S \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that

(i) given $m \in S$, $f(m, .)$ is a probability measure on $\mathcal{B}(\mathbb{R})$,
(ii) for any $E \in \mathcal{B}(\mathbb{R})$, $f(\lambda m_1 + (1-\lambda)m_2, E) = \lambda f(m_1, E) + (1-\lambda) f(m_2, E)$ whenever $\lambda \in [0, 1]$ and $m_1, m_2 \in S$

is said to be a *σ-effect function* on S.

Definition 5.1 Let L be an effect algebra. We say that a convex system S of states on L has the (E)-*property* (E as for existence) if, given a σ-effect function f on S, for any $E \in \mathcal{B}(\mathbb{R})$ there exists an element $\xi(E) \in L$ such that

(E) $f(m, E) = m(\xi(E)), m \in S$.

It was shown in [11] that the set of all σ-additive states on the effect algebra $\mathcal{E}(H)$ has the (E)-property.

Theorem 5.2 *Let L be a σ-orthocomplete effect algebra and let S be a convex order determining system of σ-additive states on L which has the* (*E*)*-property. Then for every effect function f on S, the mapping $E \mapsto \xi(E)$ from $\mathcal{B}(\mathbb{R}) \to L$ is an observable on L. Moreover, $m \mapsto f(m, .) = \Phi_\xi(m)$.*

Proof Since S is order determining, the element $\xi(E)$ is uniquely defined by property (E). Moreover, $\xi(\mathbb{R}) = 1$. Let $(E_i)_{i=1}^\infty \subseteq \mathcal{B}(\mathbb{R})$ be such that $E_i \cap E_j = \emptyset$ whenever $i \neq j$. Then for any $i \neq j$,

$$\begin{aligned}1 &\geq s(\xi(E_i \cup E_j)) = f(m, E_i \cup E_j)\\ &= f(m, E_i) + f(m, E_j) = m(\xi(E_i)) + m(\xi(E_j)),\end{aligned}$$

which implies that $\xi(E_i) \perp \xi(E_j)$ and $\xi(E_i \cup E_j) = \xi(E_i) \oplus \xi(E_j)$. By induction we prove that $\xi(\bigcup_{i=1}^n E_i) = \bigoplus_{i=1}^n \xi(E_i)$. Put $E := \bigcup_{i=1}^\infty E_i$. Then for every $m \in S$,

$$\begin{aligned}m(\xi(E)) &= f\left(m, \bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty f(m, E_i)\\ &= \lim_{n\to\infty} \sum_{i=1}^n f(m, E_i) = \lim_{n\to\infty} \sum_{i=1}^n m(\xi(E_i))\\ &= \lim_{n\to\infty} m\left(\bigoplus_{i=1}^n \xi(E_i)\right) = m\left(\bigoplus_{i=1}^\infty \xi(E_i)\right),\end{aligned}$$

where the last equality holds owing to σ-additivity of m. Since S is order determining, we obtain $\xi(E) = \bigoplus_{i=1}^\infty \xi(E_i)$. □

Theorem 5.3 *Let S be a convex order determining system of σ-additive states on a σ-orthocomplete effect algebra L such that S has* (*E*) *property. Then, given a real observable ξ on L, and a Markov kernel $\nu : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ there is a fuzzy version η of ξ.*

Proof Let ξ and ν be given. For every $m \in S$ and $E \in \mathcal{B}(\mathbb{R})$, define

$$f(m, E) := \int_{\mathbb{R}} \nu_E(x) m(\xi(dx)). \tag{19}$$

We will prove that $f : S \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is an effect function. (i): Let $m \in S$ be fixed, and let $(E_i)_{i=1}^\infty$ be a sequence of disjoint sets from $\mathcal{B}(\mathbb{R})$. Put $E = \bigcup_{i=1}^\infty E_i$. Then $\nu_E(x) := \nu(x, E) = \sum_{i=1}^\infty \nu_{E_i}(x)$, and by additivity of the integral, for every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} \nu\left(x, \bigcup_{i=1}^n E_i\right) m(\xi(dx)) = \sum_{i=1}^n \int_{\mathbb{R}} \nu(x, E_i) m(\xi(dx)).$$

Since $\nu_{\bigcup_{i=1}^n E_i} \nearrow \nu_E$ pointwise, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}} \nu\left(x, \bigcup_{i=1}^n E_i\right) m(\xi(dx)) = \int_{\mathbb{R}} \nu_E(x) m(\xi(dx)),$$

hence $f(m,E) = \sum_{i=1}^\infty f(m, E_i)$.

(ii): Let $m = \alpha m_1 + (1-\alpha)m_2$, then for every $E \in \mathcal{B}(\mathbb{R})$, $m(\xi(E)) = \alpha m_1(\xi(E)) + (1-\alpha)m_2(\xi(E))$. Therefore

$$\begin{aligned} f(m,E) &= \int_{\mathbb{R}} \nu_E(x) m(\xi(dx)) \\ &= \int_{\mathbb{R}} \nu_E(x)(\alpha m_1(\xi(dx)) + (1-\alpha) m_2(\xi(dx))) \\ &= \alpha \int_{\mathbb{R}} \nu_E(x) m_1(\xi(dx)) + (1-\alpha)\int_{\mathbb{R}} \nu_E(x) m_2(\xi(dx)) \\ &= \alpha f(m_1, E) + (1-\alpha) f(m_2, E). \end{aligned}$$

This proves that f is an effect function. Then the (E) property entails that there is an observable η such that for every $m \in S$, $E \in \mathbb{R}$, $m(\eta(E)) = \int_{\mathbb{R}} \nu(x, E) m(\xi(dx))$, that is, η is a fuzzy version of ξ. □

6 Stochastic Operators and Markov Kernels

At the beginning, we introduce some notations. Let $M(\Omega, \mathcal{A})$ denote the vector space of all complex measures on $(\Omega, \mathcal{A})$. Then $M(\Omega, \mathcal{A})$ is a Banach space with the total variation norm $\|\mu\| = |\mu|(\Omega)$.

Let μ be a σ-finite measure on $(\Omega, \mathcal{A})$, we denote by $L(\mu)$ the subspace in $M(\Omega, \mathcal{A})$ generated by all $P \in M_1^+(\Omega, \mathcal{A})$ such that $P \ll \mu$. The space $L(\mu)$ can be identified with $L_1(\Omega, \mathcal{A}, \mu)$, by extension of the map $P \mapsto \frac{dP}{d\mu}$. If $\mathcal{P} \subset M_1^+(\Omega, \mathcal{A})$, then we denote by $L(\mathcal{P})$ the subspace generated by $\bigcup_{P\in\mathcal{P}} L(P)$.

A *stochastic operator* is an affine map

$$T : \mathcal{M} \to M_1^+(\Omega_1, \mathcal{A}_1).$$

where $\mathcal{M}$ is a convex subset in $M_1^+(\Omega, \mathcal{A})$. Any stochastic operator can be extended to a positive, norm-preserving map from the Banach subspace in $M(\Omega, \mathcal{A})$ generated by $\mathcal{M}$, to $M(\Omega_1, \mathcal{A}_1)$.

Example 6.1

1. Let $\mathcal{A}_1 \subset \mathcal{A}$ be a sub-σ-algebra. Then the restriction map

$$T_{\mathcal{A}_1} : M_1^+(\Omega, \mathcal{A}) \to M_1^+(\Omega, \mathcal{A}_1),\, P \mapsto P/\mathcal{A}_1$$

is a stochastic operator.

2. Let $F:(\Omega,\mathcal{A})\to(\Omega_1,\mathcal{A}_1)$ be a measurable map. Then F defines the stochastic operator

$$T^F: M_1^+(\Omega,\mathcal{A})\to M_1^+(\Omega_1,\mathcal{A}_1),\ P\mapsto P^F,$$

where P^F is the distribution of F under P, that is,

$$P^F(B)=P(F^{-1}(B)),\ B\in\mathcal{A}_1.$$

3. Let $\nu:\Omega\times\mathcal{A}_1\to\mathbb{R}$ be a weak Markov kernel with respect to $\mathcal{P}$. Then $T_\nu: P\mapsto\nu(P)$ defines a stochastic operator $T_\nu: L(\mathcal{P})\to M(\omega_1,\mathcal{A}_1)$.

A stochastic operator is called a *statistical map* if there is a Markov kernel such that $T=T_\nu$. Note that operators $T_{\mathcal{A}_1}$ and T_F in Example 6.1 are given by Markov kernels. Indeed, if we put

$$\nu_{\mathcal{A}_1}(\omega,B)=\chi_B(\omega),\ \ \nu_F(\omega,B)=\chi_{F^{-1}(B)}(\omega), \tag{20}$$

then $T_{\mathcal{A}_1}=T_{\nu_{\mathcal{A}_1}}$, and $T_F=T_{\nu_F}$.

Proposition 6.2 [3, 4] *A stochastic operator* $T: M_1^+(\Omega,\mathcal{A})\to M_1^+(\Omega_1,\mathcal{A}_1)$ *is a statistical map if and only if for every* $B\in\mathcal{A}_1$ *there is an* $\mathcal{A}$*-measurable function* $f_B:\Omega\to[0,1]$ *such that*

$$\int_B (TP)(d\omega_1)=\int_\Omega f_B(\omega)P(d\omega) \tag{21}$$

for every $P\in M_1^+(\Omega,\mathcal{A})$.

Proof If there is a Markov kernel ν such that $T=T_\nu$, we put $f_B(\omega)=\nu(\omega,B)$.

Conversely, $\nu(\omega,B):=f_B(\omega)$ is a Markov kernel. Indeed, for every $\omega^*\in\Omega$ let δ_{ω^*} denote the corresponding Dirac measure. Equation (21) implies that $T\delta_{\omega^*}(B)=\int_\Omega f_B(\omega)\delta_{\omega^*}(d\omega)=f_B(\omega^*)$, which immediately implies the desired result. □

Let $\xi:(X,\mathcal{F})\to L$ and $\eta:(Y,\mathcal{G})\to L$ be observables such that $\xi\preceq\eta$, and let $\nu: X\times\mathcal{G}\to[0,1]$ be the corresponding confidence measure. Then the equation

$$m(\eta(G))=\int_X \nu(x,G)m(\xi(dx)),\ \ G\in\mathcal{G},\ \ m\in\mathcal{S} \tag{22}$$

can be rewritten in the form

$$\Phi_\eta=T_\nu\circ\Phi_\xi, \tag{23}$$

where T_ν is the statistical map corresponding to ν, and Φ_ξ is defined by (18).

In general, there is a little hope for a stochastic operator $T: M(\Omega,\mathcal{A})\to M(\Omega_1,\mathcal{A}_1)$ to be given by a Markov kernel (see e.g. [4]). However, for stochastic operators defined on $L(\mu)$ it is often the case.

Let us recall that the measurable space $(X,\mathcal{B})$ is a *standard Borel space* if X is a complete separable metrizable (Polish) space and $\mathcal{B}$ is the Borel σ-algebra over X.

Theorem 6.3 [28]. *Let* μ *be a* σ*-finite measure on* $(\Omega,\mathcal{A})$ *and* $(X,\mathcal{B})$ *be a standard Borel space. Let* $T: L(\mu)\to M(X,\mathcal{B})$ *be a stochastic operator. Then there is a Markov kernel* $\nu:\Omega\times\mathcal{B}\to[0,1]$ *such that* $T=T_\nu/L(\mu)$.

Let ξ be an $(\Omega, \mathcal{A})$-observable on L, and let m_0 be a faithful σ-additive state on L. Then for every $m \in \mathcal{S}$, we have $m \circ \xi \ll m_0 \circ \xi$. By Theorem 6.3, to every stochastic operator

$$T : L(m_0 \circ \xi) \to M(X, \mathcal{B}),$$

where $(X, \mathcal{B})$ is a standard Borel space, there is a Markov kernel $\nu : \Omega \times B \to [0, 1]$ such that $T = T_\nu / L(m_0 \circ \xi)$.

6.1 Coarse Graining

We keep our assumption that L is a σ-complete effect algebra and $\mathcal{S}$ is an order determining system of σ-additive states on L.

Definition 6.4 Let ξ and η be observables on L with value spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, respectively. We say that η is a *coarse graining* of ξ if there is a stochastic operator $T : M_1^+(X, \mathcal{F}) \to M_1^+(Y, \mathcal{G})$ such that $\Phi_\eta = T \circ \Phi_\xi$, where Φ_ξ is defined by (18).

Our previous discussion shows that if $\xi \preceq \eta$, then η is a coarse-graining of ξ. If η is a coarse graining of ξ, and conditions of Theorem 6.3 are satisfied, then $\xi \preceq \eta$ holds.

In particular, if $L = \mathcal{E}(H)$ (H separable), then for any observable $\xi : (\Omega, \mathcal{A}) \to L$ and any faithful state m_0, every stochastic operator $T : L(m_{0\xi}) \to M(X, \mathcal{B})$ (where $(X, \mathcal{B})$ is a standard Borel space) defines a Markov kernel ν, and hence a fuzzy version η of ξ with the confidence measure ν. If $(X, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then η is a real observable.

7 Application of the Classical Theory of Experiments to Quantum Observables

Representations of observables by probability distributions enable us to apply some results from the classical theory of experiments to quantum observables.

A (statistical) *experiment* (or *model*) is a triple $X = (\Omega, \mathcal{A}, \mathcal{P})$, where $(\Omega, \mathcal{A})$ is a measurable space and $\mathcal{P}$ is a nonempty family of measures in $M_1^+(\Omega, \mathcal{A})$. $(\Omega, \mathcal{A})$ is called the *sample space* of the experiment X.

7.1 f-Divergence

Let $P, Q \in M_1^+(\Omega, \mathcal{A})$. Recall that a *Lebesgue decomposition* of P with respect to Q is any pair (f, N), such that $f : \Omega \to \mathbb{R}$ is measurable, $f \geq 0$, $N \in \mathcal{A}$, $Q(N) = 0$, and

$$P(A) = \int_A f \, dQ + P(A \cap N), A \in \mathcal{A}.$$

The function f is called the *likelihood ratio* of P with respect to Q and denoted by $f = dP/dQ$. For example, if $P \ll Q$ and $(dP/dQ, N)$ is a Lebesgue decomposition, then $P(N) = 0$ and dP/dQ is the Radon–Nikodym derivative. If both P and Q are dominated by a σ-finite measure μ and $p = dP/d\mu$, $q = dQ/d\mu$, $N = \{q = 0\}$, then $(p/q, N)$ is a Lebesgue decomposition of P w.r. Q.

Let $P, Q \in M_1^+(\Omega, \mathcal{A})$ and let $(dP/dQ, N)$ be a Lebesgue decomposition. Let $f : [0, \infty) \to \mathbb{R}$ be a convex function. We define the *f-divergence* of P with respect to Q by [21]

$$D_f(P, Q) = \int_\Omega f\left(\frac{dP}{dQ}\right) dQ + P(N) f_\infty, \tag{24}$$

where $f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}$. It can be seen that D_f does not depend on the choice of the Lebesgue decomposition.

The f-divergence of P with respect to Q is a generalization of the total variation of $P - Q$. Choosing $f(u) := |u - 1|$ for all $u \in \mathbb{R}^+$ we obtain $D_f(P, Q) = \|P - Q\|$. More generally, if f is a strictly convex function, satisfying $f(1) = 0$, then $D_f(P, Q) \geq 0$ for all P, Q and $D_f(P, Q) = 0$ if and only if $P = Q$. In this sense, D_f can be seen as a quasi-distance in $M_1^+(\Omega, \mathcal{A})$.

For example, if $f(x) = -\log(x)$, then $D_f(P, Q)$ is the well-known I-divergence (Kullback–Leibler divergence, relative entropy)

$$I(P, Q) = \int_\Omega (\log q - \log p) q \, d\mu,$$

here μ is a dominating measure and $p = dP/d\mu$, $q = dQ/d\mu$.

Another example is the Hellinger distance

$$H(P, Q) = \frac{1}{2} \int_\Omega (p^{1/2} - q^{1/2})^2 d\mu$$

obtained by the choice $f(x) = f_H(x) = (1 - x^{1/2})$. For more examples and facts about f-divergences, see [24].

Let $\xi \preceq \eta$. The relation (9) implies that if for two states m_1, m_2 we have $m_1 \circ \xi = m_2 \circ \xi$, then also $m_1 \circ \eta = m_2 \circ \eta$ holds. That is, the discerning power of ξ with respect to states is greater than that of η. A strengthening of this result is given by the following *monotonicity theorem*. In [21, 28], it is proved for Markov kernels, but the proof works also for weak Markov kernels.

Theorem 7.1 *Let $P, Q \in M_1^+(\Omega, \mathcal{A})$ and let $\nu : \Omega \times \mathcal{A}_1 \to [0, 1]$ be a weak Markov kernel with respect to $\{P, Q\}$. Let $f : [0, \infty) \to \mathbb{R}$ be a convex function. Then*

$$D_f(P, Q) \geq D_f(\nu(P), \nu(Q)).$$

7.2 Sufficient Markov Kernels

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be an experiment, $(\Omega_1, \mathcal{A}_1)$ a measurable space and $\nu : \Omega \times \mathcal{A}_1 \to [0, 1]$ a Markov kernel. According to (12), ν assigns to every $P \in \mathcal{P}$ a measure $\nu(P) \in M_1^+(\Omega_1, \mathcal{A}_1)$ by

$$\nu(P)(A_1) = \int_\Omega \nu(x, A_1) P(dx).$$

Note that for $P, Q \in M_1^+(\Omega, \mathcal{A})$, $Q \ll P$ implies $\nu(Q) \ll \nu(P)$. Indeed, if $B \in \mathcal{A}_1$ is such that $\nu(P)(B) = \int \nu(\omega, B) dP(\omega) = 0$, then, since $\nu(\omega, B) \geq 0$, we must have $\nu(\omega, B) = 0$, P a.e. But then also $\int \nu(\omega, B) Q(d\omega) = \nu(Q)(B) = 0$.

For a measurable function $f : (\Omega, \mathcal{A}) \to [0, 1]$ and $P \in M_1^+(\Omega, \mathcal{A})$, we define the measure $f \cdot P$ as

$$f \cdot P(A) := \int_A f \, dP.$$

Then clearly $f \cdot P \ll P$, hence $\nu(f \cdot P) \ll \nu(P)$. Let us define

$$E_P^\nu(f) := d\nu(f \cdot P)/d\nu(P).$$

Definition 7.2 [21, Definition 22.1] Let $(\Omega, \mathcal{A}, \mathcal{P})$ be an experiment, $(\Omega_1, \mathcal{A}_1)$ a measurable space and $\nu : \Omega \times \mathcal{A}_1 \to [0, 1]$ a Markov kernel.

(a) ν is called *Blackwell sufficient* (for $\mathcal{P}$) if there exists a kernel $\nu' : \Omega_1 \times \mathcal{A} \to [0, 1]$ such that $\nu'(\nu(P)) = P$ holds for all $P \in \mathcal{P}$.
(b) ν is said to be *sufficient* (for $\mathcal{P}$) if to every $A \in \mathcal{A}$ there exists a measurable function $g_A : (\Omega_1, \mathcal{A}_1) \to \mathbb{R}$, such that

$$E_P^\nu(\chi_A) = g_A, \quad \nu(P) \text{ a.e. for all } P \in \mathcal{P}. \tag{25}$$

Clearly, for the observables ξ, η on L, such that $\xi \preceq \eta$, we have $\xi \sim \eta$ iff the corresponding Markov kernel ν is Blackwell sufficient for the measures $m \circ \xi$, $m \in \mathcal{S}$.

Remark 7.3 In classical statistics, sufficiency of a sub-σ-algebra $\mathcal{A}_1$ (or a statistic) for an experiment means the existence of common versions of the conditional probabilities $P(A/\mathcal{A}_1)$ for all $P \in \mathcal{P}$. The above definition is a generalization of this well-known notion: if $\mathcal{A}_1 \subset \mathcal{A}$ is a sub-σ-algebra and $\nu = \nu_{\mathcal{A}_1}$, then $E_P^\nu(\chi_A) = P(A/\mathcal{A}_1)$.

Remark 7.4 Let us fix $P \in M_1^+(\Omega, \mathcal{A})$. Let us define a map $\nu'_P : \Omega_1 \times \mathcal{A} \to \mathbb{R}$ by

$$\nu'_P(\omega_1, A) := E_P^\nu(\chi_A)(\omega_1).$$

Then we have

$$\int_A \nu(\omega, B) P(d\omega) = \int_B \nu'_P(\omega_1, A)\nu(P)(d\omega_1), \ A \in \mathcal{A}, B \in \mathcal{A}_1. \tag{26}$$

We prove that ν'_P is a weak Markov kernel with respect to $\nu(P)$.

By definition, we know that $\omega_1 \mapsto \nu'_P(\omega_1, A)$ is measurable for all $A \in \mathcal{A}$. This shows (i). Moreover, $\nu'_P(\omega_1, A) \geq 0$, $\nu(P)$ a.e. Moreover, let $B = \{\omega_1 : \nu'_P(\omega_1, A) > 1\}$ and suppose that $\nu(P)(B) > 0$. Then by (26),

$$\nu(P)(B) < \int_B \nu'_P(\omega_1, A)\nu(P)(d\omega_1) = \int_A \nu(\omega, B)P(d\omega) \leq \nu(P)(B).$$

It follows that $\nu(P)(B) = 0$, whence $\nu'_P(\omega_1, A) \leq 1$, $\nu(P)$ a.e., and (ii) is shown. We have

$$\begin{aligned} 1 \ &= \ P(\Omega) = \int_\Omega \nu(\omega, \Omega_1)P(d\omega) = \int_{\Omega_1} \nu'_P(\omega_1, \Omega)\nu(P)(d\omega_1) \\ &\Rightarrow \nu'_P(\omega_1, \Omega) = 1 \text{ a.e. } \nu(P). \end{aligned}$$

Similarly we show that $\nu'_P(\omega_1, \emptyset) = 0$ a.$e.$ $\nu(P)$ which proves (iii). Finally, let $\{A_n\}$ be a sequence in $\mathcal{A}$, such that $A_n \cap A_m = \emptyset$ if $n \neq m$. Then for $B \in \mathcal{A}_1$,

$$\begin{aligned} \int_B \nu'_P(\omega_1, \bigcup_n A_n)\nu(P)(d\omega_1) &= \int_{\bigcup_n A_n} \nu(\omega, B)P(d\omega) = \sum_n \int_{A_n} \nu(\omega, B)P(d\omega) \\ &= \int_B \sum_n \nu'_P(\omega_1, A_n)\nu(P)(d\omega_1) \end{aligned}$$

which proves (iv), so that ν'_P is indeed a weak Markov kernel with respect to $\nu(P)$. By (26)

$$\nu'_P(\nu(P)(A)) = \int_{\Omega_1} \nu'_P(\omega_1, A)\nu(P)(d\omega_1) = \int_A \nu(\omega, \Omega_1)P(d\omega) = P(A).$$

We see that sufficiency of the kernel ν, in contrast with Blackwell sufficiency, implies the existence of a *weak* Markov kernel ν', such that $\nu'(\nu(P)) = P$ holds for $P \in \mathcal{P}$.

7.3 Pairwise Sufficiency

We say that a subalgebra (Markov kernel) is *pairwise sufficient* for $\mathcal{P}$, if it is sufficient for any pair $\{P_1, P_2\}$, $P_1, P_2 \in \mathcal{P}$. Clearly, a sufficient subalgebra (Markov kernel) is pairwise sufficient. We have the following characterization of pairwise sufficient Markov kernels.

Theorem 7.5 [21, 24] (S. Kullback, R.A. Leibler, T. Csiszár). *Let* $(\Omega, \mathcal{A})$, $(\Omega_1, \mathcal{A}_1)$ *be measurable spaces*, $\nu : \Omega \times \mathcal{A}_1 \to [0, 1]$ *a Markov kernel and* $P, Q \in M_1^+(\Omega, \mathcal{A})$. *Then the following are equivalent.*

(i) ν *is sufficient for* $\{P, Q\}$.
(ii) *For any convex function* f *on* $\mathbb{R}^+$ *one has*

$$D_f(\nu(P), \nu(Q)) = D_f(P, Q). \tag{27}$$

(iii) *There is a strictly convex function* f *on* $\mathbb{R}^+$ *such that*

$$D_f(\nu(P), \nu(Q)) = D_f(P, Q) < \infty. \tag{28}$$

Note that we may take the Hellinger distance $H(P, Q)$ in (iii).

7.4 Dominated Families

Let $\mathcal{P} \subset M_1^+(\Omega, \mathcal{A})$. We say that $\mathcal{P}$ is a *dominated family*, if there is a σ-finite measure μ such that $\mathcal{P} \ll \mu$. If this is the case, then we can find a finite measure μ_0, dominating $\mathcal{P}$. It is clear that if $\mathcal{P} \ll \mu$, then we have $C(\mathcal{P}) \ll \mu$, where

$$C(\mathcal{P}) = \left\{ \sum_n \lambda_n P_n : \lambda_n \geq 0, \sum_n \lambda_n = 1, P_n \in \mathcal{P} \right\}.$$

If we also have $\mu(A) = 0$ whenever $P(A) = 0$ for all $P \in \mathcal{P}$, then we write $\mathcal{P} \sim \mu$.

Lemma 7.6 [18] *Let* $\mathcal{P}$ *be a dominated family. Then there is a convex combination* $P_0 = \sum_n \lambda_n P_n$ *of elements of* $P_n \in \mathcal{P}$, $n \in \mathbb{N}$, *such that* $\mathcal{P} \sim P_0$.

The following theorem is well known.

Theorem 7.7 [18, 28] *Let* $\mathcal{P} \subset M_1^+(\Omega, \mathcal{A})$ *be a dominated family and let* $P_0 \in C(\mathcal{P})$ *be such that* $\mathcal{P} \sim P_0$. *Let* $\mathcal{A}_1 \subset \mathcal{A}$ *be a sub-*σ*-algebra. Then the following are equivalent.*

(i) $\mathcal{A}_1$ *is sufficient for* $\mathcal{P}$.
(ii) $\mathcal{A}_1$ *is pairwise sufficient for* $\mathcal{P}$.
(iii) $\mathcal{A}_1$ *is sufficient for the pair* $\{P, P_0\}$ *for every* $P \in \mathcal{P}$.

A similar statement holds also for Markov kernels. Since the proof is not easy to find in the literature, we give it here. We will first show that we can describe sufficient Markov kernels in terms of sufficient subalgebras.

Let $\nu : \Omega \times \mathcal{A}_1 \to [0,1]$ be a weak Markov kernel with respect to $\mathcal{P} \subset M_1^+(\Omega, \mathcal{A})$. For $P \in L(\mathcal{P}) \cap M_1^+(\Omega, \mathcal{A})$ we define a probability measure $P \times \nu \in M_1^+(\Omega \times \Omega_1, \mathcal{A} \otimes \mathcal{A}_1)$, by

$$P \times \nu(A \times B) = \int_A \nu(\omega, B) P(d\omega), \ \ A \in \mathcal{A}, \ \ B \in \mathcal{A}_1. \tag{29}$$

Note that we have $P \times \nu(\Omega \times B) = \nu(B)$ and $P \times \nu(A \times \Omega_1) = P(A)$ for $A \in \mathcal{A}$, $B \in \mathcal{A}_1$.

Lemma 7.8 [21] *Let $\mathcal{P} \subset M_1^+(\Omega, \mathcal{A})$ and let $\nu : \Omega \times \mathcal{A}_1 \to [0,1]$ be a Markov kernel. Then ν is sufficient for $\mathcal{P}$ if and only if the sub-σ-algebra $\mathcal{A}_0 = \{\emptyset, \Omega\} \otimes \mathcal{A}_1 \subset \mathcal{A} \otimes \mathcal{A}_1$ is sufficient for $\{P \times \nu : P \in \mathcal{P}\}$.*

Proof Let ν be sufficient and let ν' be the corresponding weak Markov kernel, see Remark 7.4. For $A \in \mathcal{A}$, $B \in \mathcal{A}_1$, we define a function $f_{A\times B} : \Omega \times \Omega_1 \to [0,1]$ by

$$f_{A\times B}(\omega, \omega_1) = \nu'(\omega_1, A)\chi_B(\omega_1).$$

It is clear that $f_{A\times B}$ is $\mathcal{A}_0$-measurable, moreover, for $B_1 \in \mathcal{A}_1$ and $P \in \mathcal{P}$,

$$\begin{aligned}\int_{\Omega\times B_1} f_{A\times B} d(P\times\nu) &= \int_{B\cap B_1} \nu'(\omega_1, A)\nu(P)(d\omega_1) = \int_A \nu(\omega, B\cap B_1)P(d\omega)\\ &= P\times\nu(A\times B\cap\Omega\times B_1).\end{aligned}$$

It follows that $f_{A\times B}$ is the common version of the conditional probability $f_{A\times B} = P \times \nu(A \times B/\mathcal{A}_0)$, $P \times \nu$ a.e., for all $P \in \mathcal{P}$.

Conversely, suppose that $\mathcal{A}_0$ is sufficient for $\{P \times \nu : P \in \mathcal{P}\}$ and let $f_{A\times B} = P \times \nu(A \times B/\mathcal{A}_0)$, $P \times \nu$ a.e. for all $P \in \mathcal{P}$. Then, since $f_{A\times B}$ is $\mathcal{A}_0$-measurable, it depends only from ω_1. Put $\nu'(\omega_1, A) = f_{A\times\Omega_1}(\omega_1)$, then for $B \in \mathcal{A}_1$ and $P \in \mathcal{P}$,

$$\int_B \nu'(\omega_1, A)\nu(P)(d\omega_1) = \int_{\Omega\times B} f_{A\times\Omega_1} d(P\times\nu) = P\times\nu(A\times B) = \int_A \nu(\omega, B)P(d\omega),$$

so that $\nu' = \nu'_P$, $\nu(P)$-a.e. □

Theorem 7.9 *Let $\mathcal{P} \subset M_1^+(\Omega, \mathcal{A})$ be dominated and let $P_0 \in C(\mathcal{P})$ be such that $\mathcal{P} \sim P_0$. Let $\nu : \Omega \times \mathcal{A}_1 \to [0,1]$ be a Markov kernel. Then ν is sufficient for $\mathcal{P}$ if and only if ν is sufficient for $\{P, P_0\}$ for every $P \in \mathcal{P}$.*

Proof Let us denote

$$\mathcal{P} \times \nu := \{P \times \nu : P \in \mathcal{P}\}.$$

By Lemma 7.8, ν is sufficient for all $\{P, P_0\}$ if and only if the sub-σ-algebra $\mathcal{A}_0$ is sufficient for all $\{P \times \nu, P_0 \times \nu\}$. Clearly, $P_0 \times \nu \in C(\mathcal{P} \times \nu)$ and if we have $\mathcal{P} \times \nu \sim P_0 \times \nu$, then, by Theorem 7.7, $\mathcal{A}_0$ is sufficient for $\mathcal{P} \times \nu$ and therefore ν is sufficient for $\mathcal{P}$. It is enough to prove that $\{P \times \nu : P \in \mathcal{P}\}$ is dominated by $P_0 \times \nu$.

For this, fix $\epsilon > 0$ and let $A \in \mathcal{A}$, $B \in \mathcal{A}_1$. By Kolmogorov inequality,

$$P_0 \times \nu(A \times B) = \int_A \nu(\omega, B) P_0(d\omega) \geq k P_0(A \cap \{\nu(\omega, B) \geq k\}) \tag{30}$$

for all $k \geq 0$, moreover, since $\nu(\omega, B) \leq 1$,

$$P \times \nu(A \times B) = \int_A \nu(\omega, B) P(d\omega) = \int_{A \cap \{\nu(\omega B) \geq k\}} \nu(\omega, B) P(d\omega) + \int_{A \cap \{\nu(\omega, B) < k\}} \nu(\omega, B) P(d\omega) \leq P(A \cap \{\nu(\omega, B) \geq k\}) + k.$$

Since $P \ll P_0$, there is some $\delta > 0$, such that $P(A) < \epsilon/2$ if $P_0(A) < 2\delta/\epsilon$. Put $k = \epsilon/2$ in (30), then $P_0 \times \nu(A \times B) < \delta$ implies that $P_0 \cap \{\nu(\omega, B) \geq \epsilon/2\}) < 2\delta/\epsilon$ and $P \times \nu(A \times B) < \epsilon$.

Let now $C \in \mathcal{A} \otimes \mathcal{A}_1$ be such that $P_0 \times \nu(C) < \delta/2$. Then, since C can be approximated by rectangles, there are some $A \in \mathcal{A}$ and $B \in \mathcal{A}_1$, such that $A \times B \supset C$ and $P_0 \times \nu(A \times B) < \delta$. This implies that $P(C) \leq P \times \nu(A \times B) < \epsilon$. □

A comparison of Blackwell sufficiency and sufficiency is given in the following theorem ([21, Theorem 22.11]).

Theorem 7.10 *Let $(\Omega, \mathcal{A}, \mathcal{P})$ be an experiment, $(\Omega_1, \mathcal{A}_1)$ a measurable space and $\nu : \Omega \times \mathcal{A}_1 \to [0, 1]$ a Markov kernel.*

(i) *If $(\Omega, \mathcal{A}, \mathcal{P})$ is μ-dominated by a σ-finite measure μ on $(\Omega, \mathcal{A})$ and ν is Blackwell sufficient, then ν is sufficient.*
(ii) *If $(\Omega_1, \mathcal{A}_1, \nu(\mathcal{P}))$ is μ_1-dominated by a σ-finite measure μ_1, $(\Omega, \mathcal{A})$ is a standard Borel space and ν is sufficient for $\mathcal{P}$, then ν is also Blackwell sufficient.*

Proof (i) can be proved from Theorems 7.1, 7.5 and 7.9, (ii) follows from Remark 7.4 and Theorem 6.3. □

We can list the results of the present section as follows.

Corollary 7.11 *Let $\mathcal{P} \subset M_1^+(\Omega, \mathcal{A}_1)$ be a dominated family and let $P_0 \in C(\mathcal{P})$ be such that $\mathcal{P} \sim P_0$. Let $\nu : \Omega \times \mathcal{A}_1 \to [0, 1]$ be a Markov kernel. Then the following are equivalent.*

(i) *ν is pairwise sufficient for $\mathcal{P}$.*
(ii) *ν is sufficient for $\{P, P_0\}$ for each $P \in \mathcal{P}$.*
(iii) *For all $P \in \mathcal{P}$ and all convex functions $f : [0, \infty) \to \mathbb{R}$, we have*

$$D_f(P, P_0) = D_f(\nu(P), \nu(P_0)).$$

(iv) *There is a strictly convex function $f : [0, \infty) \to \mathbb{R}$, such that for all $P \in \mathbb{P}$*

$$D_f(P, P_0) = D_f(\nu(P), \nu(P_0)) < \infty.$$

(v) *ν is sufficient for $\mathcal{P}$.*

(vi) *There is a weak Markov kernel* $\nu' : \Omega_1 \times \mathcal{A}_1 \to [0,1]$ *with respect to* $\{\nu(P) : P \in \mathcal{P}\}$, *such that* $\nu'(\nu(P)) = P$ *for all* $P \in \mathcal{P}$. *If* $(\Omega, \mathcal{A})$ *is a standard Borel space, then* ν' *is a Markov kernel and* ν *is Blackwell sufficient for* $\mathcal{P}$.

Proof The equivalence of (i)–(v) follows directly from our previous results, (v)$\Longrightarrow$(vi) follows from the Remark 7.4. The implication (vi)$\Longrightarrow$(v) follows from Theorems 7.1 and 7.5. □

7.5 Application to Fuzzy Quantum Observables

Recall that if m_0 is a faithful state on L then, for every state m and every observable ξ on L, it holds $m \circ \xi \ll m_0 \circ \xi$. Therefore $\mathcal{P} = \{m \circ \xi : m \in \mathcal{S}\}$ is a dominated family, with $\mathcal{P} \sim m_0 \circ \xi$. Applying Corollary 7.11 and Theorem 7.10, we obtain the following theorem.

Theorem 7.12 *Let* L *be a* σ*-orthocomplete effect algebra with an order determining system* $\mathcal{S}$ *of* σ*-additive states,* ξ *and* η *be real observables on* L *such that* $\xi \preceq \eta$ *with a confidence measure* ν, *and let there exist a faithful state* $m_0 \in \mathcal{S}$. *The following conditions are equivalent.*

(i) ν *is pairwise sufficient for* $\{m \circ \xi : m \in \mathcal{S}\}$.
(ii) ν *is sufficient for* $\{m \circ \xi, m_0 \circ \xi\}$ *for all* $m \in \mathcal{S}$.
(iii) *for all* $m \in \mathcal{S}$ *and all convex functions* $f : [0,\infty) \to \mathbb{R}$, *we have*

$$D_f(m \circ \xi, m_0 \circ \xi) = D_f(\nu(m \circ \xi), \nu(m_0 \circ \xi)).$$

(iv) *There is a strictly convex function* $f : [0,\infty) \to \mathbb{R}$ *such that for all* $m \in \mathcal{S}$

$$D_f(m \circ \xi, m_0 \circ \xi) = D_f(\nu(m \circ \xi), \nu(m_0 \circ \xi)) < \infty.$$

(v) ν *is sufficient for* $\{m \circ \xi : m \in \mathcal{S}\}$.
(vi) ν *is Blackwell sufficient for* $\{m \circ \xi : m \in \mathcal{S}\}$.
(vii) $\xi \sim \eta$.

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